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Vortex dynamics in nonrelativistic Abelian Higgs model

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ABSTRACT

The dynamics of the gauge vortex with arbitrary form of a contour is considered in the framework of the nonrelativistic Abelian Higgs model, including the possibility of the gauge field interaction with the fermion asymmetric background. The equations for the time derivatives of the curvature and the torsion of the vortex contour generalizing the Betchov–Da Rios equations in hydrodynamics, are obtained. They are applied to study the conservation of helicity of the gauge field forming the vortex, twist, and writhe numbers of the vortex contour. It is shown that the conservation of helicity is broken when both terms in the equation of the vortex motion are present, the first due to the exchange of excitations of the phase and modulus of the scalar field and the second one due to the coupling of the gauge field forming the vortex, with the fermion asymmetric background.

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1. Introduction

The string-like (or vortex) solutions of the field-theoretical models are widely discussed in application to numerous physical systems, from the really observed in condensed matter physics (quantized vortices in He⁴ [1], Abrikosov lines in type II superconductors [2], vortices in Bose–Einstein condensates [3], etc.) to hypothetical cosmic strings [4]. Of particular interest are the situations when the dynamics of the vortex contours can be deduced from the field equations of the underlying field theory. Different types of models are invoked to study the dynamics. In particular, the dynamics of cosmic strings was studied in the framework of the Nambu–Goto action [5,6]. The latter was shown to result [7] from the relativistic version of the Abelian Higgs model (AHM). The solution in the form of the static cosmic string was obtained from the static version of AHM [9]. Another limiting case is the nonrelativistic one, which can be deduced from the nonrelativistic Abelian Higgs model (NRAHM). The static limit of this model coincides with the Ginzburg–Landau theory [8].

The vortex in the models with the local gauge symmetry is called the local one. The equation of the nonrelativistic motion of the local vortex, in the idealized situation of the negligible dissipation, was obtained in the work [10] in the framework of NRAHM.

The feature of the equation of the vortex motion in the model considered in Ref. [10] is that except for the usual term stating that the vortex ring velocity points to the direction of bi-normal, there appears the additional term arising due to the exchange of the excitations of the phase and modulus of the scalar field between different segments of the contour. Yet the model considered in [10] is not the most general one. As was pointed out in [11], there could be another term in the gauge vortex equation of motion in case when the gauge field is coupled to the asymmetric background of chiral fermions [12–16]. In the static case, the influence of this additional term on the form of the vortex contour was considered in Ref. [11].

Another interesting aspect of the time-dependent gauge vortex solutions is that they can elucidate the interrelations among such characteristics of the closed contours as the helicity of gauge field forming the vortex, the writhe and torsion (twist) numbers and their possible dependence on time [18–20]. In case of pure hydrodynamics, the corresponding equation was deduced in Ref. [21]. Note that the problem of dynamical evolution of the vortex filaments has a long story traced to the beginning of the twentieth century [21]. The corresponding equations for the time derivatives of the curvature and torsion were rediscovered, in particular, in Refs. [22,23]. The historical review of these developments is presented in Ref. [24].

The aim of the present work is to consider the dynamical evolution of the gauge vortex string in NRAHM, including the possibility of its interaction with the static fermion asymmetric background.

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2. Nonrelativistic Abelian Higgs model with the gauge vortex

Nonrelativistic Abelian Higgs model incorporating gauge vortices is given by the following Lagrangian density [10]:

$$\begin{aligned} \mathcal{L} = & \frac{1}{8\pi} (\mathbf{E}^2 - \mathbf{H}^2) - \frac{g}{2} (|\psi|^2 - n_0)^2 \\ & + \frac{1}{2} [\psi^* (i\hbar \partial_t - q\varphi + qa_0) \psi + \text{c.c.}] \\ & - \frac{1}{2m} \left| \left(-i\hbar \nabla - \frac{q}{c} \mathbf{A} + \frac{q}{c} \mathbf{a} \right) \psi \right|^2 - \rho_0 \varphi, \end{aligned} \quad (1)$$

where $\mathbf{E} = -\partial_t \mathbf{A}/c - \nabla \varphi$ and $\mathbf{H} = \nabla \times \mathbf{A}$ are the electric and magnetic field strengths, the four-vector gauge potential is $A_\mu = (\varphi, \mathbf{A})$; q, m are the charge, mass, of the particles forming the condensate of the scalar field, c is the velocity of light. The quantities n_0 and ρ_0 are, respectively, the density of the scalar field condensate and the homogeneous positive charge density introduced to provide the net neutrality of the system:

$$\rho_0 + qn_0 = 0. \quad (2)$$

The coupling constant g can be related to the sound velocity c_s . See Eq. (7) below. The limiting case of this model, with the gauge coupling constant set to zero, corresponds to the model of Gross–Pitaevskii [25,26] possessing the solutions in the form of global vortices observed in Bose systems like HeII or BEC. The vortex is represented as the line of singular phase χ_s of the scalar field $\psi = \sqrt{n_0} e^{i\chi_s}$ which forms the spatially homogeneous condensate of the density n_0 everywhere except for the vortex core, where it goes to zero at the transverse distances of the order of the healing (or correlation) length ξ .

The notations in Eq. (1) are as follows. The four-vector

$$a_\mu = -\frac{\hbar c}{q} \partial_\mu \chi_s$$

is the four-gradient of the singular phase,

$$[\nabla \times \nabla] \chi_s = 2\pi \int d\sigma \mathbf{X}' \delta^{(3)}(\mathbf{x} - \mathbf{X}), \quad (3)$$

serving as the source of the gauge vortex with the unit flux quantum whose space–time location is given by the vector $\mathbf{X} \equiv \mathbf{X}(t, \sigma)$. The length of the contour, $l = \int_{\sigma_i}^{\sigma_f} |\mathbf{X}'| d\sigma$, is the natural choice of σ . The explicit expressions for a_μ found from Eq. (3), in the gauge $\nabla \mathbf{a} = 0$, are:

$$\begin{aligned} a_0 &= \frac{\hbar}{2q} \oint \frac{([\dot{\mathbf{X}} \times \mathbf{X}'], \mathbf{x} - \mathbf{X})}{|\mathbf{x} - \mathbf{X}|^3} d\sigma, \\ \mathbf{a} &= \frac{\hbar c}{2q} \oint \frac{[\mathbf{X}' \times (\mathbf{x} - \mathbf{X})]}{|\mathbf{x} - \mathbf{X}|^3} d\sigma. \end{aligned} \quad (4)$$

Hereafter, the prime (dot) means the differentiation with respect to the contour parameter σ (time), and $[\mathbf{a} \times \mathbf{b}]$ $[(\mathbf{a}, \mathbf{b})]$ stands for the vector (scalar) product of two vectors \mathbf{a} and \mathbf{b} .

The term $\propto g$ in Eq. (1) can be rewritten in the form

$$-\frac{g}{2} (|\psi|^2 - n_0)^2 = -\frac{g}{2} |\psi|^4 + gn_0 |\psi|^2 - \frac{g}{2} n_0^2,$$

demonstrating that the first term in the right hand side describes the s-wave interaction between the bosons forming the condensate, the second one is $\mu |\psi|^2$, with the chemical potential $\mu = gn_0$ pertinent for the Bose gas with the s-wave interaction [27], and the third one is irrelevant constant. After integration over space and going to the Hamiltonian this second term goes to $-\mu N$ thus showing that the actual Hamiltonian is in fact $H' = H - \mu N$ [27].

This chemical potential μ is due to the background of the bosonic condensate. The equation of motion of the gauge vortex resulting from NRAHM was shown [10] to look like

$$\begin{aligned} [\dot{\mathbf{X}} \times \mathbf{X}'] &= \frac{\hbar}{2m} \left(\mathbf{X}'' \ln \frac{\lambda_L}{\xi} + \frac{1}{c_s^2} \left[\left(\frac{\partial}{\partial t} [\dot{\mathbf{X}} \times \mathbf{X}'] \right) \times \mathbf{X}' \right] \ln \frac{\lambda_s}{\xi} \right), \end{aligned} \quad (5)$$

where

$$\lambda_L = \sqrt{\frac{mc^2}{4\pi n_0 q^2}} \quad (6)$$

is the London penetration depth,

$$c_s = \sqrt{\frac{n_0 g}{m}} \quad (7)$$

is the velocity of sound [28],

$$\xi = \frac{\hbar}{2mc_s} \quad (8)$$

is the healing length (the characteristic size of the vortex core). The term $\propto 1/c_s^2$ and the intermediate scale

$$\lambda_s = \lambda_L \frac{c_s}{c} \ll \lambda_L, \quad (9)$$

as was shown in Ref. [10], are generated dynamically, due to the exchange of the fluctuations of the phase and the modulus of the scalar field ψ between the different segments of the gauge vortex. The term $[\dot{\mathbf{X}} \times \mathbf{X}']$ originates from the variation of the term $\int q |\psi|^2 a_0 dt d^3x$ in the action [10]. For further use let us introduce also the quantum of magnetic flux

$$\Phi_0 = \frac{2\pi \hbar c}{q}. \quad (10)$$

It is known [12,13,15] that the presence of the background of chiral fermions (whose asymmetry is characterized by the chemical potential μ_f) coupled to the gauge field A_μ (in particular, the Abelian one [15,16]), induces the term $-\mu_f N_{CS}$ in the Hamiltonian, where

$$N_{CS} = \frac{1}{\Phi_0^2} \int d^3x (\mathbf{A}, [\nabla \times \mathbf{A}]) \quad (11)$$

is the Chern–Simons number carried by the Abelian gauge field of the vortex. Coupling of the Abelian gauge field with the chiral fermion [15] takes place, in particular, in the standard model of particle physics [14] where the right electron is coupled to the hypermagnetic gauge field. As is seen from Eq. (11), the Chern–Simons number in the Abelian case is proportional to helicity,

$$h_A = \int d^3x (\mathbf{A}, [\nabla \times \mathbf{A}]). \quad (12)$$

Using the expressions for \mathbf{A} and $\mathbf{B} = [\nabla \times \mathbf{A}]$ from Ref. [10],

$$\begin{aligned} \mathbf{A}(\mathbf{x}, t) &= \Phi_0 \int \frac{d^3k}{(2\pi)^3} \frac{i/\lambda_L^2}{\mathbf{k}^2 (\mathbf{k}^2 + 1/\lambda_L^2)} \times \oint d\sigma [\mathbf{k} \times \mathbf{X}'] e^{i(\mathbf{k}, \mathbf{x} - \mathbf{X})}, \\ \mathbf{H}(\mathbf{x}, t) &= \Phi_0 \int \frac{d^3k}{(2\pi)^3} \frac{1/\lambda_L^2}{\mathbf{k}^2 + 1/\lambda_L^2} \times \oint d\sigma \mathbf{X}' e^{i(\mathbf{k}, \mathbf{x} - \mathbf{X})}, \end{aligned} \quad (13)$$

one obtains the expression for the helicity of the single vortex with the unit flux quantum and its variation over \mathbf{X} :

$$\begin{aligned}
h_A &= i\Phi_0^2 \oint d\sigma_1 d\sigma_2 \int \frac{d^3k}{(2\pi)^3} \frac{1/\lambda_L^2}{\mathbf{k}^2 + 1/\lambda_L^2} \left(\frac{1/\lambda_L^2}{\mathbf{k}^2 + 1/\lambda_L^2} \right)^2 \\
&\quad \times \left(\mathbf{k}, \left[\frac{\partial \mathbf{X}(\sigma_1)}{\partial \sigma_1} \times \frac{\partial \mathbf{X}(\sigma_2)}{\partial \sigma_2} \right] \right) \exp[-i(\mathbf{k}, \mathbf{X}(\sigma_1) - \mathbf{X}(\sigma_2))], \\
\delta h_A &= 2\Phi_0^2 \int \frac{d^3k}{(2\pi)^3} \left(\frac{1/\lambda_L^2}{\mathbf{k}^2 + 1/\lambda_L^2} \right)^2 \\
&\quad \times \oint d\sigma_1 d\sigma_2 (\delta \mathbf{X}_1, [\mathbf{X}'_1 \times \mathbf{X}'_2]) e^{-i(\mathbf{k}, \mathbf{X}_{12})} \\
&= \frac{\Phi_0^2}{4\pi\lambda_L^3} \oint d\sigma_1 d\sigma_2 (\delta \mathbf{X}_1, [\mathbf{X}'_1 \times \mathbf{X}'_2]) e^{-|\mathbf{X}_{12}|/\lambda_L}, \quad (14)
\end{aligned}$$

where $\mathbf{X}_{1,2} \equiv \mathbf{X}(\sigma_{1,2})$, $\mathbf{X}_{12} = \mathbf{X}_1 - \mathbf{X}_2$. Since the segments of the vortex contour located at the distances greater than λ_L give exponentially damped contribution, one can use the expansion in $z = \sigma_2 - \sigma_1$,

$$\mathbf{X}(\sigma + z) = \mathbf{X}(\sigma) + z\mathbf{X}'(\sigma) + \frac{z^2}{2}\mathbf{X}''(\sigma) + \frac{z^3}{6}\mathbf{X}'''(\sigma) + \dots \quad (15)$$

and keep only the first non-vanishing term surviving in the limit $\lambda_L \rightarrow \infty$:

$$\begin{aligned}
\delta h_A &= \frac{\Phi_0^2}{8\pi\lambda_L^3} \oint d\sigma ([\mathbf{X}' \times \mathbf{X}'''], \delta \mathbf{X}) \int_{-\infty}^{\infty} z^2 e^{-|z|/\lambda_L} dz \\
&= \frac{\Phi_0^2}{2\pi} \oint d\sigma ([\mathbf{X}' \times \mathbf{X}'''], \delta \mathbf{X}). \quad (16)
\end{aligned}$$

The variation of the corresponding term in the action, $\Delta S = \mu_f \int dt N_{CS}$, gives the term $\propto [\mathbf{X}' \times \mathbf{X}''']$ in the equation of motion [11] which now looks like

$$\begin{aligned}
[\dot{\mathbf{X}} \times \mathbf{X}'] &= \gamma \mathbf{X}'' + T_0 \left[\left(\frac{\partial}{\partial t} [\dot{\mathbf{X}} \times \mathbf{X}'] \right) \times \mathbf{X}' \right] \\
&\quad + \tilde{\mu} [\mathbf{X}' \times \mathbf{X}'''], \quad (17)
\end{aligned}$$

where the following shorthand notations are used:

$$\begin{aligned}
\gamma &= \frac{\hbar}{2m} \ln \frac{\lambda_L}{\xi}, \\
T_0 &= \frac{\hbar}{2mc_s^2} \ln \frac{\lambda_s}{\xi}, \\
\tilde{\mu} &= \frac{\mu_f}{4\pi^2 \hbar n_0}. \quad (18)
\end{aligned}$$

The term $\gamma \mathbf{X}''$ is due to the tension which pushes the vortex segment to the direction of the normal vector \mathbf{n} . Other terms are discussed below. When both T_0 and $\tilde{\mu}$ are set to zero, the known relation $\dot{\mathbf{X}} = \gamma \kappa \mathbf{b}$ is recovered. In this case the static solution with zero velocity is possible for the straight vortex for which $\kappa = 0$, $\tau = 0$. The particular dynamical solution to the latter equation describes the vortex ring moving with the velocity $V = \gamma \kappa$ in the direction orthogonal to the ring plane. If $T_0 = 0$, $\tilde{\mu} \neq 0$ then the nontrivial static solution in the form of the helix with $\kappa = \text{const}$ and $\tau = \gamma/\tilde{\mu}$ is possible [11].

Taking into account the fact that the physical motion of the vortex is orthogonal to the unit vector \mathbf{X}' , one can rewrite Eq. (17) in the form of the wave equation with the additional terms:

$$\frac{1}{c_0^2} \ddot{\mathbf{X}} - \mathbf{X}'' + \frac{1}{\gamma} [\dot{\mathbf{X}} \times \mathbf{X}'] + \frac{\tilde{\mu}}{\gamma} [\mathbf{X}' \times \mathbf{X}'''] = 0, \quad (19)$$

where

$$c_0 = \left(\frac{\gamma}{T_0} \right)^{1/2} = c_s \left(\frac{\ln \lambda_L/\xi}{\ln \lambda_s/\xi} \right)^{1/2}. \quad (20)$$

The interpretation of the additional terms is the following. The term $\propto [\dot{\mathbf{X}} \times \mathbf{X}']$ is the analog of the Magnus force acting on the moving vortex due to the nonzero circulation of the supercurrent. In the classical hydrodynamics, in the direction orthogonal to the displacement velocity, the velocity of the circulating flow adds with the displacement velocity on one side, and subtracts from it on the opposite side. By Bernoulli's equation, the pressure in the liquid is greater in the regions where the velocity is lower, yielding the force in the direction orthogonal to the vectors $\dot{\mathbf{X}}$ and \mathbf{X}' . Of course, in the case of our interest the situation is much more intricate because one should take into account the role of a number of quantum excitations, but the qualitative picture is probably the same.

The term $\propto [\mathbf{X}' \times \mathbf{X}''']$ is induced by the anomaly. Indeed, the variation of the induced Chern–Simons term in the action over vector potential \mathbf{A} results in the anomalous current $\mathbf{j}_{an} \propto \mu_f \mathbf{H}$ pointed along the direction of the magnetic field. This relation is the corner stone of the widely discussed chiral magnetic effect. See the review [17] and references therein. This current is subjected to the action of the Lorentz force $\propto [\mathbf{j}_{an} \times \mathbf{H}]$ from the magnetic field produced by the nearby segments of the gauge vortex. Taking into account the exponential damping of the magnetic field strength and the expansion (15), one can write the leading contribution to the anomalous part of the Lorentz force per unit length:

$$\begin{aligned}
\mathbf{f}_L &\propto \mu_f \int_{-\infty}^{\infty} dz [\mathbf{X}'(\sigma) \times \mathbf{X}'(\sigma + z)] e^{-|z|/\lambda_L} \\
&\propto \mu_f [\mathbf{X}' \times \mathbf{X}''']. \quad (21)
\end{aligned}$$

So, the discussed term is the macroscopic manifestation of the quantum anomaly. The helical form of the vortex contour in the static case [11] is due to the balance between the tension and the anomaly-induced Lorentz force.

3. Time derivatives of the basic contour characteristics

As is known, the derivatives over the contour parameter σ of the basic unit contour vectors of normal \mathbf{n} , bi-normal \mathbf{b} , and tangent \mathbf{X}' constituting the right triple, $\mathbf{X}' = [\mathbf{n} \times \mathbf{b}]$, are given by the Frenet–Serret equations

$$\frac{\partial}{\partial \sigma} \begin{pmatrix} \mathbf{X}' \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{X}' \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix}, \quad (22)$$

where κ , τ are the curvature and torsion, respectively. One can write down the analogous set of equations for the time derivatives of the above vectors taking into account the fact that they are unit ones and that the physical motions of the contour are transverse (i.e. orthogonal to \mathbf{X}'):

$$\frac{\partial}{\partial t} \begin{pmatrix} \mathbf{X}' \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix} = \begin{pmatrix} 0 & a_{Xn} & a_{Xb} \\ a_{nX} & 0 & a_{nb} \\ a_{bX} & a_{bn} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{X}' \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix} \equiv \hat{A} \begin{pmatrix} \mathbf{X}' \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix}. \quad (23)$$

Equating the commutator of the derivatives over contour parameter and time to zero one finds that the matrix \hat{A} in (23) is anti-symmetric, $a_{bn} = -a_{nb}$, $a_{nX} = -a_{Xn}$, and the following equations are valid:

$$\begin{aligned} a_{nb} &= (\tau a_{Xn} + a'_{Xb})/\kappa, \\ \dot{\kappa} &= a'_{Xn} - \tau a_{Xb}, \\ \dot{\tau} &= \kappa a_{Xb} + a'_{nb}, \end{aligned} \quad (24)$$

so that

$$\hat{A} = \begin{pmatrix} 0 & a_{Xn} & a_{Xb} \\ -a_{Xn} & 0 & a_{nb} \\ -a_{Xb} & -a_{nb} & 0 \end{pmatrix}. \quad (25)$$

One can relate $\dot{\kappa}$ and $\dot{\tau}$ with the components of the vortex velocity $\dot{\mathbf{X}}$, or, equivalently, with the dual vector $\mathbf{W} = [\dot{\mathbf{X}} \times \mathbf{X}] = W_n \mathbf{n} + W_b \mathbf{b}$, $\dot{\mathbf{X}} = W_n \mathbf{b} - W_b \mathbf{n}$. Differentiating the definition of the vector \mathbf{W} over σ and using the Frenet–Serret equations and Eq. (23), one obtains

$$\frac{\partial \mathbf{X}'}{\partial t} = \mathbf{b}(W'_n - \tau W_b) - \mathbf{n}(W'_b + \tau W_n). \quad (26)$$

This equation permits one to make the identification

$$\begin{aligned} a_{Xn} &= -W'_b - \tau W_n, \\ a_{Xb} &= W'_n - \tau W_b, \end{aligned} \quad (27)$$

so that

$$\begin{aligned} \dot{\kappa} &= -(W'_b + \tau W_n)' - \tau(W'_n - \tau W_b), \\ \dot{\tau} &= \kappa(W'_n - \tau W_b) \\ &\quad + \{[(W'_n - \tau W_b)' - \tau(W'_b + \tau W_n)]/\kappa\}'. \end{aligned} \quad (28)$$

With these ingredients one can rewrite Eq. (17) in terms of the components of \mathbf{W} :

$$\begin{aligned} W_n &= \kappa(\gamma - \tilde{\mu}\tau) + T_0 \{ \dot{W}_b \\ &\quad + [(W'_n - \tau W_b)' - \tau(W'_b + \tau W_n)] W_n / \kappa \}, \\ W_b &= \tilde{\mu}\kappa' - T_0 \{ \dot{W}_n \\ &\quad - [(W'_n - \tau W_b)' - \tau(W'_b + \tau W_n)] W_b / \kappa \}. \end{aligned} \quad (29)$$

The system of equations (28) is in fact the result of definitions which are not related to the specific dynamical model. The latter is provided by Eq. (29). In particular, when T_0 and $\tilde{\mu}$ both vanish, $W_n = \gamma\kappa$, $W_b = 0$, one finds in this limiting case that

$$\begin{aligned} \dot{\kappa} &= -\gamma(2\kappa'\tau + \kappa\tau'), \\ \dot{\tau} &= \gamma \left[\kappa\kappa' + \left(\frac{\kappa''}{\kappa} - \tau^2 \right)' \right]. \end{aligned} \quad (30)$$

After the re-scaling $\kappa \rightarrow \gamma^{-1/2}\kappa$, $\tau \rightarrow \gamma^{-1/2}\tau$, $\sigma \rightarrow \gamma^{1/2}\sigma$ the parameter γ drops from equations, and one gets the Betchov–Da Rios equations [21–23] known in hydrodynamics for a long time.

Equations (29) are nonlinear, so let us find their solution which is exact in the fermion chemical potential $\tilde{\mu}$ but perturbative to the first order in T_0 . The zeroth order solution is

$$\begin{aligned} W_n^{(0)} &= \kappa(\gamma - \tilde{\mu}\tau), \\ W_b^{(0)} &= \tilde{\mu}\kappa'. \end{aligned} \quad (31)$$

The corresponding analog of the Betchov–Da Rios equations found from (28) looks as

$$\begin{aligned} \dot{\kappa}^{(0)} &= -(\gamma - \tilde{\mu}\tau)(2\kappa'\tau + \kappa\tau') - \tilde{\mu}(\kappa'' - \kappa\tau^2), \\ \dot{\tau}^{(0)} &= \kappa[\gamma\kappa' - \tilde{\mu}(2\kappa'\tau + \kappa\tau')] \\ &\quad + \left\{ \left[(\gamma - \tilde{\mu}\tau)(\kappa'' - \kappa\tau^2) + \tilde{\mu}(2\kappa'\tau + \kappa\tau')' \right] / \kappa \right\}'. \end{aligned} \quad (32)$$

It is straightforward to obtain the first order correction $\propto T_0$ using the expressions

$$\begin{aligned} W_n^{(1)} &= T_0 \left\{ \dot{W}_b + \frac{1}{\kappa} [(W'_n - \tau W_b)' - \tau(W'_b + \tau W_n)] W_n \right\}^{(0)}, \\ W_b^{(1)} &= -T_0 \left\{ \dot{W}_n \right. \\ &\quad \left. - \frac{1}{\kappa} [(W'_n - \tau W_b)' - \tau(W'_b + \tau W_n)] W_b \right\}^{(0)}, \end{aligned} \quad (33)$$

but the corresponding expressions are excessively long. Instead, one can find the time derivative of the total torsion. From the third equation in (24) and the second equation in (27) one obtains

$$\oint \dot{\tau} d\sigma = \oint \kappa a_{Xb} d\sigma = - \oint (\kappa' W_n + \kappa \tau W_b) d\sigma. \quad (34)$$

Inserting here Eqs. (31) and (33) and using Eq. (32), after some lengthy algebraic manipulations and integrations by parts, one finds

$$\oint \dot{\tau} d\sigma = \frac{1}{4} \gamma T_0 \tilde{\mu} \oint \kappa^4 \tau' d\sigma. \quad (35)$$

This expression shows that the total torsion of the gauge vortex in the nonrelativistic Abelian Higgs model is not conserved in case of the presence of coupling of the gauge field forming the vortex, with the fermion asymmetric background described by $\tilde{\mu} \neq 0$. Coming back to the physical quantities one obtains

$$\oint \dot{\tau} d\sigma = \frac{\hbar \mu_f}{64\pi^2 m^2 c_s^2 n_0} \times \ln \frac{\lambda_L}{\xi} \times \ln \frac{\lambda_s}{\xi} \oint \kappa^4 \tau' d\sigma. \quad (36)$$

One should have in mind that the equation of motion (5) was obtained in the paper [10] in the London limit $\lambda_L/\xi \gg 1$, $\lambda_s/\xi \gg 1$, so Eq. (36) tells us that the time derivative of the total torsion, being proportional to the small factor $1/c_s^2$, is small but not negligible, because it is multiplied by two large logarithmic factors. Note also that the coherence length ξ characterizes the radius of the vortex core where the condensate of the scalar field vanishes.

The time derivative of helicity is obtained from Eq. (16):

$$\begin{aligned} \dot{h}_A &= \frac{\Phi_0^2}{2\pi} \oint d\sigma ([\dot{\mathbf{X}} \times \mathbf{X}'], \mathbf{X}''') \\ &= \frac{\Phi_0^2}{2\pi} \oint d\sigma (\mathbf{W}, \kappa' \mathbf{n} + \kappa \tau \mathbf{b}) \\ &= \frac{\Phi_0^2}{2\pi} \oint d\sigma (\kappa' W_n + \kappa \tau W_b) \\ &= -\frac{\Phi_0^2}{2\pi} \oint \dot{\tau} d\sigma \\ &\equiv -\Phi_0^2 \frac{d\text{Tw}}{dt}, \end{aligned} \quad (37)$$

where Tw is the twist number,

$$\text{Tw} = \frac{1}{2\pi} \oint \tau d\sigma = \frac{1}{2\pi} \oint (\mathbf{X}', [\mathbf{n}' \times \mathbf{n}]) d\sigma. \quad (38)$$

One can relate the above quantities with the writhe number. The term $([\mathbf{X}' \times \mathbf{X}'''], \delta \mathbf{X})$ in the integrand of expression for the variation of helicity (16), hence the corresponding term $([\mathbf{X}' \times \dot{\mathbf{X}}], \mathbf{X}''')$ in the expressions for the time derivatives (37), has the geometrical meaning. Indeed, let us find the writhe and its variation:

$$\begin{aligned}
W\Gamma &= \frac{1}{4\pi} \oint d\sigma_1 \oint d\sigma_2 \frac{(\mathbf{X}_{12}, [\mathbf{X}'_1 \times \mathbf{X}'_2])}{|\mathbf{X}_{12}|^3} \\
&= -i \int \frac{d^3k}{(2\pi)^3} \oint d\sigma_1 d\sigma_2 \frac{(\mathbf{k}, [\mathbf{X}'_2 \times \mathbf{X}'_1])}{k^2} \times e^{i\mathbf{k} \cdot \mathbf{X}_{21}}, \\
\delta W\Gamma &= -i \int \frac{d^3k}{(2\pi)^3} \frac{\mathbf{k}}{k^2} \oint d\sigma_1 d\sigma_2 \{ [\delta \mathbf{X}'_2 \times \mathbf{X}'_1] + [\mathbf{X}'_2 \times \delta \mathbf{X}'_1] \\
&\quad + i[\mathbf{X}'_2 \times \mathbf{X}'_1](\mathbf{k} \cdot \mathbf{X}_{21}) \} e^{i\mathbf{k} \cdot \mathbf{X}_{21}} \\
&= \int \frac{d^3k}{(2\pi)^3} \frac{\mathbf{k}}{k^2} \oint d\sigma_1 d\sigma_2 \{ -[\delta \mathbf{X}_2 \times \mathbf{X}'_1](\mathbf{k} \cdot \mathbf{X}_{21}) \\
&\quad + [\mathbf{X}'_2 \times \delta \mathbf{X}_1](\mathbf{k} \cdot \mathbf{X}'_1) + [\mathbf{X}'_2 \times \mathbf{X}'_1](\mathbf{k} \cdot \delta \mathbf{X}_2) \\
&\quad - [\mathbf{X}'_2 \times \mathbf{X}'_1](\mathbf{k} \cdot \delta \mathbf{X}_1) \} e^{i\mathbf{k} \cdot \mathbf{X}_{21}} \\
&= \int \frac{d^3k}{(2\pi)^3} \frac{\mathbf{k}}{k^2} \oint d\sigma_1 d\sigma_2 \{ [\mathbf{X}'_2 \times [\mathbf{k} \times [\delta \mathbf{X}_1 \times \mathbf{X}'_1]] \\
&\quad - [\mathbf{X}'_1 \times [\mathbf{k} \times [\mathbf{X}_2 \times \delta \mathbf{X}'_2]]] \} e^{i\mathbf{k} \cdot \mathbf{X}_{21}} \\
&= \int \frac{d^3k}{(2\pi)^3} \oint d\sigma_1 d\sigma_2 (\delta \mathbf{X}_{21} \cdot [\mathbf{X}'_2 \times \mathbf{X}'_1]) e^{i\mathbf{k} \cdot \mathbf{X}_{21}} \\
&= \oint d\sigma_1 d\sigma_2 (\delta \mathbf{X}_{21} \cdot [\mathbf{X}'_2 \times \mathbf{X}'_1]) \delta^{(3)}(\mathbf{X}_{21}). \tag{39}
\end{aligned}$$

The ambiguity $0 \times \infty$ in the integrand can be regularized with the help of expression

$$\delta^{(3)}(\mathbf{X}) = \lim_{\xi \rightarrow 0} \frac{e^{-\mathbf{X}^2/2\xi^2}}{(2\pi)^{3/2}\xi^3}.$$

Then, using the expansion (15) in $z = \sigma_2 - \sigma_1$, the regularized expression for the variation of the writhe number can be written in the form:

$$\begin{aligned}
\delta W\Gamma_{\text{reg}} &= \lim_{\xi \rightarrow 0} \frac{1}{(2\pi)^{3/2}\xi^3} \oint d\sigma_1 d\sigma_2 e^{-\mathbf{X}_{21}^2/2\xi^2} (\delta \mathbf{X}_{21}, [\mathbf{X}'_2 \times \mathbf{X}'_1]) \\
&= \lim_{\xi \rightarrow 0} \frac{1}{(2\pi)^{3/2}2\xi^3} \oint d\sigma \int_{-\infty}^{\infty} dz z^2 e^{-z^2/2\xi^2} \\
&\quad \times (\delta \mathbf{X}, [\mathbf{X}' \times \mathbf{X}''']) \\
&= \frac{1}{2\pi} \oint d\sigma (\delta \mathbf{X}, [\mathbf{X}' \times \mathbf{X}''']) \\
&= \delta h_A / \Phi_0^2, \tag{40}
\end{aligned}$$

that is

$$\frac{dW\Gamma}{dt} = \frac{\dot{h}_A}{\Phi_0^2}. \tag{41}$$

Combining this equation and Eq. (37) one can see that

$$\frac{d}{dt}(W\Gamma + T\omega) = 0, \tag{42}$$

i.e., the sum of the twist and writhe is conserved as it should [18–20,29,30]. In the meantime, the helicity of the gauge field configuration in the model considered here is not conserved. It is essential that the conservation of helicity is broken due to the fact that both terms in the equation of the vortex motion are present, the first due to the exchange of excitations of the phase and modulus of the scalar field resulting in the factor $\propto c_s^{-2}$, and the second one due to the coupling of the gauge field forming the vortex, with the fermion asymmetric background $\propto \mu_f$. The latter serves as the source and sink of the helicity. Such coupling is in fact the consequence of the U(1) anomaly of the current of the chiral fermions [12]. This anomaly is a quantum phenomenon hence the appearance of the Planck constant in the right hand side of Eq. (36).

4. Discussion and conclusion

The vortex solutions with nontrivial dynamics in the gauge field models like the nonrelativistic Abelian one considered here, are interesting in that they allow to apply some general mathematical concepts like the topology of curves and the equations for the time derivatives of the curvature and torsion, to the situations beyond the earlier considered hydro- or electrodynamical ones [18–20]. As is shown in the present work, one can relate the above time derivatives with the vortex velocity. The latter, in turn, is governed by the equation of motion specific for a given underlying model. In the presence of both the dynamical correction to the equation of motion due to exchange of the fluctuations of the phase and modulus of the scalar field, and the coupling with the fermion asymmetric background, the helicity of the gauge field is not conserved. Contrary to the situation in magnetohydrodynamics where the helicity conservation is violated by the dissipative effects (finite conductivity of the medium), in the present work, the nonconservation is reversible, because both signs in the right hand side of Eq. (36) are possible.

As for the application to physics, one can hardly hope to observe the closed or curved vortices in type II superconductors immersed into the static external magnetic fields. However, their production could be quite possible in the fast temperature quench [31]. See, for example, Ref. [32], where such a mechanism was considered in the framework of the 2D Abelian Higgs model. Also, numerous simulations of the phase transitions in the early Universe point to a considerable portion of the closed cosmic strings [4]. The type of the model considered in the present work is one where the charged particles interacting with the gauge field, condense uniformly except for the vortex core. The total neutrality is provided by some static background of the opposite charge. The cosmic strings of such type are not excluded in the models of the dark (hidden) sector [33,34]. However, the relation of the present work parameters q , n_0 , and g with ones from the models of the hidden sector, goes beyond the scope of this paper.

The global vortices, i.e. the vortices in the models with the global U(1) symmetry, of arbitrary form, were observed in the dilute Bose or/and Fermi gases. They are described by the limiting case of the equation of motion (5) in which one should replace the lengths λ_L and λ_s by the single parameter R whose magnitude is of the order of the size of the vessel or the size of the cloud of the condensed atoms [10]. Of course, the term $\propto \mu_f$ is excluded in this case because gauge field is absent, hence $\oint \dot{\tau} d\sigma = 0$. In this case, the torsion and writhe are conserved separately.

References

- [1] R.J. Donnelly, *Quantized Vortices in Helium II*, Cambridge University Press, 1991.
- [2] A.A. Abrikosov, Zh. Èksp. Teor. Fiz. 32 (1957) 1442.
- [3] K.W. Madison, F. Chavy, W. Wohlleben, J. Dalibard, Phys. Rev. Lett. 84 (2000) 806.
- [4] A. Vilenkin, E.P.S. Shellard, *Cosmic Strings and Other Topological Defects*, Cambridge University Press, 1994.
- [5] Y. Nambu, Lectures at the Copenhagen Symposium, 1970.
- [6] J. Scherk, Rev. Mod. Phys. 47 (1975) 123.
- [7] D. Förster, Nucl. Phys. B 81 (1974) 84.
- [8] V.L. Ginzburg, L.D. Landau, Zh. Èksp. Teor. Fiz. 20 (1950) 1064.
- [9] H.B. Nielsen, P. Olesen, Nucl. Phys. B 61 (1973) 45.
- [10] A.A. Kozhevnikov, Int. J. Mod. Phys. B 24 (2010) 605.
- [11] A.A. Kozhevnikov, Phys. Lett. B 461 (1999) 256.
- [12] A.N. Redlich, L.C.R. Wijewardhana, Phys. Rev. Lett. 54 (1985) 970.
- [13] V.A. Rubakov, A.N. Tavkhelidze, Theor. Math. Phys. 65 (1985) 250; V.A. Rubakov, A.N. Tavkhelidze, Phys. Lett. B 165 (1985) 109.
- [14] M. Joyce, M.E. Shaposhnikov, Phys. Rev. Lett. 79 (1997) 1193.
- [15] M. Laine, J. High Energy Phys. 0510 (2005) 056.
- [16] V.B. Semikoz, J.W.F. Valle, J. Cosmol. Astropart. Phys. 11 (2011) 048.
- [17] D.E. Kharzeev, Prog. Part. Nucl. Phys. 75 (2014) 133.

- [18] H.K. Moffatt, *J. Fluid Mech.* 35 (1969) 117.
- [19] M.A. Berger, G.B. Field, *J. Fluid Mech.* 147 (1984) 133.
- [20] H.K. Moffatt, R.L. Ricca, *Proc. R. Soc. Lond. Ser. A* 439 (1992) 411.
- [21] L.S. Da Rios, *Rend. Circ. Mat. Palermo* 22 (1906) 117.
- [22] R. Betchov, *J. Fluid Mech.* 22 (1965) 471.
- [23] H. Hasimoto, *J. Fluid Mech.* 51 (1965) 477.
- [24] R.L. Ricca, *Nature* 352 (1991) 561.
- [25] L.P. Pitaevskii, *Sov. Phys. JETP* 13 (1961) 451.
- [26] E.P. Gross, *Nuovo Cimento* 20 (1961) 454.
- [27] E.M. Lifshits, L.P. Pitaevskii, *Statistical Physics, Part 2*, Nauka Publishers, 1978.
- [28] N.N. Bogolyubov, *J. Phys. (USSR)* 11 (1947) 23.
- [29] F.B. Fuller, *Proc. Natl. Acad. Sci. USA* 68 (1971) 815.
- [30] W.F. Pohl, *J. Math. Mech.* 17 (1968) 975.
- [31] W.H. Zurek, *Phys. Rep.* 276 (1996) 177.
- [32] G.J. Stephens, Luís M.A. Bettencourt, W.H. Zurek, *Phys. Rev. Lett.* 88 (2002) 137004.
- [33] T. Vachaspati, *Phys. Rev. D* 80 (2009) 063502.
- [34] J.M. Hyde, A.J. Long, T. Vachaspati, *Phys. Rev. D* 89 (2014) 065031.